DERIVED EQUIVALENT CONJUGATE K3 SURFACES

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Abstract. We show that there exist a complex projective K3 surface X and an automorphism $\sigma \in \mathrm{Aut}(\mathbb{C})$ such that the conjugate K3 surface X^σ is a non-isomorphic Fourier–Mukai partner of X.

1. Introduction

By definition two complex projective K3 surfaces X and Y are called Fourier–Mukai partners (or derived equivalent) if there exists a $\mathbb{C}$-linear equivalence between their derived categories of coherent sheaves. Due to results by Mukai ([14]) and Orlov ([17]) we have geometric and cohomological criteria for K3 surfaces to be derived equivalent. In particular, it follows that any given K3 surface has only finitely many non-isomorphic Fourier–Mukai partners. It is also possible to view derived equivalent K3 surfaces as elements in the orbit of the action of a certain discrete group on the moduli space of (generalized) K3 surfaces (see e.g. [10]).

There is a different group acting on the moduli space, namely the group of automorphisms of the complex numbers. The action is defined as follows:

Consider a complex projective K3 surface X and $\sigma \in \mathrm{Aut}(\mathbb{C})$. Then we define the *conjugate K3 surface* X^σ by the fiber product

$$\begin{tikzcd} X^\sigma \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "\sigma^*"] & \mathrm{Spec}(\mathbb{C}). \end{tikzcd}$$

The K3 surfaces X and X^σ will in general be non-isomorphic as schemes over $\mathbb{C}$ but clearly they are isomorphic over $\mathbb{K} = \mathbb{Q}$ and thus there is a $\mathbb{Q}$-linear equivalence $\mathrm{D^b}(X) \simeq_{\mathbb{Q}} \mathrm{D^b}(X^\sigma)$. An obvious question is whether one can find examples where there is a $\mathbb{C}$-linear equivalence as well, without X and X^σ being isomorphic over $\mathbb{C}$.

Thus, we would like to understand the connection between the actions of $\mathrm{Aut}(\mathbb{C})$ and the above mentioned discrete group on the moduli space of projective K3 surfaces.

The main result is the following theorem which shows that the orbits of the two group actions can intersect in more than one point:

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Theorem *There exists a complex projective K3 surface X and an automorphism $\sigma \in \mathrm{Aut}(\mathbb{C})$ such that the conjugate K3 surface X^σ is a non-isomorphic complex K3 surface, but there exists a $\mathbb{C}$-linear equivalence $\mathrm{D^b}(X) \xrightarrow{\sim} \mathrm{D^b}(X^\sigma)$.*

The basic idea of the proof is to find a curve C in the moduli space of K3 surfaces over $\overline{\mathbb{Q}}$ which is invariant under a so called Mukai involution, which is a map sending a K3 surface X to a certain moduli space $\mathrm{M}_X(v)$ of stable sheaves on X. The induced automorphism of the function field $K(C)$ will allow us to produce two non-isomorphic derived equivalent K3 surfaces over $\mathbb{C}$. Since this automorphism of $K(C)$ extends to an automorphism of $\mathbb{C}$ these two K3 surfaces will also be conjugate.
The note is organized as follows: First, we consider moduli spaces of K3 surfaces over $\overline{\mathbb{Q}}$. Then we define Mukai involutions on these moduli spaces and study their fixed point locus in a special case. This information enables us to construct the above mentioned curve. In the last section it is shown that a similar construction method works, and a similar result holds, for abelian surfaces.
A small appendix provides an alternative proof of one of the results in section 3.

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2. K3 surfaces over $\overline{\mathbb{Q}}$ and their moduli spaces

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. A *K3 surface* is a smooth two-dimensional projective variety X over $\mathbb{K}$ with trivial canonical bundle and $H^1(X, \mathcal{O}_X) = 0$. An example of such a surface is given by a smooth quartic in $\mathbb{P}^3_{\mathbb{K}}$, e.g. the Fermat quartic $\left\{\underline{x} \in \mathbb{P}^3_{\mathbb{K}} : x_0^4 + \ldots + x_3^4 = 0\right\}$.

An ample line bundle L on X will be called a *polarization*. The self-intersection number (L, L) of such an L is called the *degree* of the polarized K3 surface (X, L). A polarization is *primitive* if L is not a power of any line bundle on X. Using Serre duality and the Riemann–Roch theorem, it is easy to see that (L, L) is even for any line bundle L on X. Further, if L is ample it is effective, its Hilbert polynomial is given by $h_L(t) = (\frac{1}{2}(L, L))t^2 + 2$, $\mathcal{L}^n$ is generated by global sections for $n \geq 2$ and is very ample for $n \geq 3$ (for the latter results see [19]).

A *family of K3 surfaces* is a proper and flat morphism $\pi : \mathcal{X} \longrightarrow S$ over a scheme S, which as all schemes in the following will be assumed to be of finite type, such that the geometric fibers of π are K3 surfaces. A family of (primitively) polarized K3 surfaces is given by a map π as above together with a line bundle $\mathcal{L}$ on $\mathcal{X}$ which defines a (primitive) polarization restricted to each geometric fiber. Note that $\mathcal{L}^n$ is relatively very ample over S for $n \geq 3$.

Using the general theory in [24] it is possible to construct a coarse quasi-projective separated moduli scheme $\mathcal{M}_{2d}^{\mathbb{K}}$ for primitively polarized K3 surfaces of degree $2d$ over $\mathbb{K}$ as follows: Consider the Hilbert scheme Hilb_N^P representing subvarieties of $\mathbb{P}_{\mathbb{K}}^N$ with Hilbert polynomial $P(x) := n^2dx^2 + 2$, where $n \geq 3$, d is a natural number (which should be thought of as $(\frac{1}{2}\mathcal{L}^2)$) and $N = P(1) - 1$. Then there exists on open subscheme U of Hilb_N^P representing primitively polarized K3 surfaces together with an embedding into $\mathbb{P}_{\mathbb{K}}^N$ and the GIT quotient $U/PGL(N+1)$ is the coarse moduli scheme in question.

For $\mathbb{K} = \mathbb{C}$ there is a different construction (which uses lattice and Hodge theory), which shows that the moduli scheme of polarized complex K3 surfaces $\mathcal{M}_{2d}^{\mathbb{C}}$ is actually a 19-dimensional reduced, irreducible and normal space. For details see [1].

We would like to see that $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$ inherits all the above mentioned properties. A first step is the following

Lemma 2.1. $\mathcal{M}_{2d}^{\mathbb{C}} \simeq \mathcal{M}_{2d}^{\overline{\mathbb{Q}}} \times_{\mathrm{Spec}(\overline{\mathbb{Q}})} \mathrm{Spec}(\mathbb{C})$.

Proof. Both schemes are constructed as GIT-quotients and this is compatible with field extensions, cf. [16, Prop. 1.14]. □

Using the lemma we can now deduce

Proposition 2.2. *$\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$ is an integral 19-dimensional scheme.*

Proof. Since $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}} \times_{\mathrm{Spec}(\overline{\mathbb{Q}})} \mathrm{Spec}(\mathbb{C})$ is irreducible, so is $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$ being its image under the projection. Further, $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$ is reduced, since $\mathcal{M}_{2d}^{\mathbb{C}}$ is and we can check this property using an affine cover. The statement about the dimension is clear. □

We would also like to control the singularities of $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$. This is given by

Proposition 2.3. *$\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$ is a normal scheme.*

Proof. We will use the following result from commutative algebra:
Let R and S be two $\mathbb{K}$-algebras such that $R \otimes_{\mathbb{K}} S$ is Noetherian. Then $R \otimes_{\mathbb{K}} S$ is normal if and only if R and S are normal (this is a special case of [23, Theorem 1.6]).

Since normality is a local property, we may take an open affine subset $\mathrm{Spec}(A)$ in $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$. Then, by normality of $\mathcal{M}_{2d}^{\mathbb{C}}$, we have that $A \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ is normal (and of course Noetherian) and hence A is. □

3. Mukai involutions

In this section we will study so called Mukai involutions on $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$. We will use several results obtained in [21] where the action of the group of Mukai involutions on $\mathcal{M}_{2d}^{\mathbb{C}}$ was investigated.

For a complex $2d$-porarized K3 surface (X, L) consider the moduli space $M_X = M_X(v)$ of stable sheaves on X whose numerical invariants are given by a Mukai vector $v \in H^0(X,\mathbb{Z}) \oplus H^2(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z})$. We will only consider vectors of the form $v = (r, L, s)$ fulfilling (a) $L^2 = 2rs = 2d$, (b) $\gcd(r, s) = 1$ and (c) $r \leq s$. These choices ensure that M_X is a fine moduli space and a $2d$-polarized K3 surface. We will use the same notation over $\overline{\mathbb{Q}}$. In this case M_X is also in $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$, which follows from the fact that this is true over $\mathbb{C}$ and that any line bundle on the K3 surface $(M_X)_{\mathbb{C}}$ is already defined over $\overline{\mathbb{Q}}$. We define the *Mukai involution* g to be the map sending X to M_X.

Proposition 3.1. *The Mukai involution g is a morphism.*

Proof. Consider the universal family $f : \mathcal{X} \longrightarrow U$, where U is the above used open subscheme of the Hilbert scheme. Note that U is reduced. The morphism f is projective and hence there exists a relative moduli space $\mathcal{M}(v) \longrightarrow U$ such that over $t \in U$ we have the moduli space $M_{\mathcal{X}_t}(v)$ (compare [11, Theorem 4.3.7]). We know that the polarization $\widetilde{\mathcal{L}}$, which exists by construction, has the property that its intersection number on the fibers is a quadratic multiple, say a, of the given degree $2d$, which one can see by first looking at K3 surfaces of Picard rank 1 and then using that the intersection number is (locally) constant. Now, the étale sheafification of the relative Picard functor is representable by a scheme $\mathrm{Pic}_{\mathcal{M}(v)/U}$ (see e.g. [5, Chapter 8]) and the image of the morphism $f : U \longrightarrow \mathrm{Pic}_{\mathcal{M}(v)/U}$ defined by $\widetilde{\mathcal{L}}$ lies in the image of the map $[a] : \mathrm{Pic}_{\mathcal{M}(v)/U} \longrightarrow \mathrm{Pic}_{\mathcal{M}(v)/U}$ (where $[a]$ is the multiplication by a). We therefore have a commutative diagram

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[dr, "\widetilde{f}"'] & \mathrm{Pic}_{\mathcal{M}(v)/U} \\ & \mathrm{Pic}_{\mathcal{M}(v)/U} \arrow[u, "{[a]}"'] \end{tikzcd}$$

The morphism $\widetilde{f}$ defines an element of the étale Picard functor, which by definition is represented by a line bundle $\mathcal{L}'$ on the fiber product $\mathcal{M}(v)' := \mathcal{M}(v) \times_U U'$, for some étale covering $\pi : U' \longrightarrow U$, with the property that $\widetilde{\pi}^*(\widetilde{\mathcal{L}}) \simeq \mathcal{L}'^a$ ($\widetilde{\pi}$ is the natural projection). Thus, $\mathcal{M}(v)' \longrightarrow U'$ is a family of K3 surfaces with a polarization of degree $2d$ and we therefore get a map $\alpha : U' \longrightarrow \mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$. Using descent theory described in [8, Exposé VIII] we know that there exists a morphism $\beta : U \longrightarrow \mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$ such that $\beta\pi = \alpha$ if and only if α commutes with the two projections from $U' \times_U U'$, i.e. $\alpha p_1 = \alpha p_2$. But the latter condition is clear for closed, and hence for all (everything is reduced), points by the fact that α is the classifying map of the family and the K3 surfaces over the closed points of a fibre of π are all isomorphic. Thus we have a map $\beta : U \longrightarrow \mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$ sending t to $M_{\mathcal{X}_t}(v)$. Since α is equivariant, it descends to a morphism h from the GIT-quotient $U/PGL = \mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$ to $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$

so that the following diagram commutes

$$\begin{tikzcd} U \arrow[r, "\beta"] \arrow[d] & \mathcal{M}_{2d}^{\overline{\mathbb{Q}}} \\ \mathcal{M}_{2d}^{\overline{\mathbb{Q}}} \arrow[ur, dotted, "h"] & \end{tikzcd}$$

By definition we have that $g = h$ and thus g is a morphism. □

Remark 3.2. Clearly the same proof works for $\mathcal{M}_{2d}^{\mathbb{C}}$ (and any other algebraically closed field). In particular, we shall need $g_{\mathbb{C}} : \mathcal{M}_{2d}^{\mathbb{C}} \longrightarrow \mathcal{M}_{2d}^{\mathbb{C}}$.

We can consider a Mukai involution from a different point of view using the following result (see [14]):

Theorem (Mukai, Orlov) Let X and Y be two complex projective K3 surfaces. The following conditions are equivalent:
(a) X and Y are Fourier–Mukai partners.
(b) Y is a fine moduli space of stable sheaves on X.

It follows from this theorem that $X_{\mathbb{C}} := X \times_{\mathrm{Spec}\overline{\mathbb{Q}}} \mathrm{Spec}(\mathbb{C})$ and $M_X \times_{\mathrm{Spec}\overline{\mathbb{Q}}} \mathrm{Spec}(\mathbb{C}) = M_{X_{\mathbb{C}}}(v)$ are FM–partners.

Remark 3.3. If a complex K3 surface X has Picard rank one all Fourier–Mukai partners can be determined explicitly by describing the Mukai vectors v as above. Furthermore, $M_X(v) \not\cong M_X(v')$ for $v \neq v'$.

From now on we will consider the case $2d = 12$. In this case there is only one Mukai involution sending a K3 surface X to the moduli space $M_X = M(2, l, 3)$. This involution will be denoted by g. The first step is to invesigate its fixed point locus. It was proved in [21] that over $\mathbb{C}$ there is precisely one divisor D in the fixed point locus of $g_{\mathbb{C}}$. We will now prove the

Proposition 3.4. *The fixed point locus Fix(g) of the morphism* $g : \mathcal{M}_{2d}^{\overline{\mathbb{Q}}} \longrightarrow \mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$ *contains a divisor.*

Proof. The fixed point locus of a morphism can be defined as the intersection of the diagonal and the graph which are both defined over $\overline{\mathbb{Q}}$. Since this construction commutes with base change we have that $\mathrm{Fix}(g_{\mathbb{C}}) = \mathrm{Fix}(g) \times_{\mathrm{Spec}(\overline{\mathbb{Q}})} \mathrm{Spec}(\mathbb{C})$. Therefore Fix($g$) has to contain a divisor. □

Remark 3.5. The divisor in Fix($g_{\mathbb{C}}$) corresponds to K3 surfaces whose Picard lattice contains a certain rank two nondegenerate even lattice. From [12, Corollary 2.5 and Corollary 1.9] we know that there exists a K3 surface X of Picard rank 20 in D. By [20, Theorem

6] X can be defined over a number field and in particular over $\overline{\mathbb{Q}}$. In fact, the points of D defined over $\overline{\mathbb{Q}}$ are dense in D which gives another proof of the above proposition.

4. A special curve in $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$

We want to construct an irreducible g-invariant curve in $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$. The strategy is rather simple: Take a curve in the quotient space $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}/\langle \mathrm{id}, g\rangle$ and pull it back to $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$. The curve C we get will clearly be g-invariant. The irreducibility will be achieved by using Bertini's theorem.
We first recall the following

Proposition 4.1. *The quotient $\mathcal{K} = \mathcal{M}_{2d}^{\overline{\mathbb{Q}}}/\langle \mathrm{id}, g\rangle$ is an algebraic variety. The projection map $\pi : \mathcal{M}_{2d}^{\overline{\mathbb{Q}}} \longrightarrow \mathcal{K}$ is finite and surjective.*

Proof. Since $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$ is quasi-projective, this follows from [15], pp. 66–69. □

To ensure that C is irreducible, we want it to be connected and regular. The first property is proved by the following

Proposition 4.2. *Let A be an irreducible curve in $\mathcal{K}$ which is not contained in the image of the fixed point locus of g but intersects it in at least one point. Then the pullback curve $C = \pi^{-1}(A) = \mathcal{M}_{2d}^{\overline{\mathbb{Q}}} \times_{\mathcal{K}} A$ is connected. Furthermore g acts non-trivially on C.*

Proof. Assume the converse, then there exist disjoint closed non-empty subsets $C = W_1 \bigsqcup W_2$. Since A intersects the fixed point locus, there exists a point $x \in A$ whose reduced fiber is precisely one point y. We may assume that $y \in W_1$. Since π is finite and thus proper, $\pi(W_1), \pi(W_2)$ are closed in A. Since A is irreducible and both sets are non-empty, we must have $\pi(W_1) = \pi(W_2)$. However, this is impossible, since $\pi(W_2)$ does not contain x. The last assertion is obvious. □

Next we have to make sure that C is regular. To do this we use Bertini's theorem: Let A be given as an intersection of hyperplanes. If these hyperplanes are generic, then A is regular away from the singularities of $\mathcal{K}$. In order to control these we need the following

Lemma 4.3. *Let R be a normal integral domain, let H be a finite group of automorphisms of R and let R^H be the ring of invariants. Then R^H is normal.*

Proof. If $z \in Q(R^H)$ is integral over R^H, it is also integral over R and hence $z \in R$, since R is normal. On the other hand, $h(z) = z$ for all $h \in H$ and therefore $z \in R^H$. Thus, R^H is normal. □

Since we know $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$ to be normal, it follows from the lemma that $\mathcal{K}$ is normal as well. Thus, if A is generic among those curves intersecting the fixed point locus in a generic point

of the divisor, it will be regular and the same will hold for C (cf. [9, III, Cor. 10.9]). Thus, we have proved

Proposition 4.4. *There exists a g-invariant connected and regular, hence irreducible, curve C (on which g acts non-trivially) in $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$.* □

However, $\mathcal{M}_{2d}^{\overline{\mathbb{Q}}}$ is just a coarse moduli space and thus a priori we do not have a family over C. This problem is avoided by the following

Proposition 4.5. *There exists a family $\mathcal{X} \longrightarrow C'$ over an irreducible curve C' such that the classifying map of this family is a finite surjective morphism $C' \longrightarrow C$. Furthermore, the inclusion of function fields $K(C) \longrightarrow K(C')$ is a Galois extension.*

Proof. The first statement is well-known, cf. e.g. [24, Theorem 9.25] (the general idea is that the wanted curve lives in the Hilbert scheme). Denote the covering curve by $\widetilde{C}$. Considering function fields we have a finite (and of course separable) extension $\mathbb{K} := K(C) \longrightarrow \mathbb{L} := K(\widetilde{C})$. If this extension is not normal, we can take the normal closure $\mathbb{L}'$ of $\mathbb{L}$ to get a Galois extension $\mathbb{K} \longrightarrow \mathbb{L}'$. Geometrically this just corresponds to a finite surjective morphism from a new curve $C' \longrightarrow \widetilde{C}$ and a (pullback-)family over C'. Hence the result. □

The morphism g induces an automorphism of $\mathbb{K}$ which can be lifted to an automorphism $\widetilde{g}$ of $\overline{\mathbb{K}} = \overline{\mathbb{L}}$ (see e.g. [25, Theorem 6]). Considering the composition of $\widetilde{g}$ with the inclusion $\mathbb{L} \longrightarrow \overline{\mathbb{K}}$ and using the normality of $\mathbb{L}$, we see that we get an automorphism g' of $\mathbb{L}$ which clearly extends g. This automorphism then gives an automorphism of the curve C'. Since g' is an extension of g, it has the same geometric interpretation, namely sending a fibre X to a K3 surface which is isomorphic to $M_X(v)$.

The geometric fibre of the generic point of C' is a K3 surface and therefore the generic fibre itself is as well (use [9, III, Prop. 9.3]). Denote this K3 surface over $\mathbb{L}$ as $X_{\mathbb{L}}$. Base change via g' gives a second K3 surface over $\mathbb{L}$ which by construction is $\mathrm{M}_{X_{\mathbb{L}}}(2, l, 3) =: X'_{\mathbb{L}}$. Now fix an imbedding i of $\mathbb{L}$ into $\mathbb{C}$. Denoting the induced action of g' on $\mathrm{Spec}(\mathbb{L})$ by g'^{*}, we have the following diagram:

$$\begin{array}{ccccccc}
X_{\mathbb{C}} & \longrightarrow & X_{\mathbb{L}} & \longleftarrow & X'_{\mathbb{L}} & \longleftarrow & X'_{\mathbb{C}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Spec}(\mathbb{C}) & \xrightarrow{i^*} & \mathrm{Spec}(\mathbb{L}) & \xleftarrow{g'^*} & \mathrm{Spec}(\mathbb{L}) & \xleftarrow{i^*} & \mathrm{Spec}(\mathbb{C}).
\end{array}$$

Clearly $X'_{\mathbb{C}} = \mathrm{M}_{X_{\mathbb{C}}}(2, l, 3)$ and thus $X'_{\mathbb{C}}$ is not isomorphic to $X_{\mathbb{C}}$ for a generic (i.e. of Picard rank 1) $X_{\mathbb{C}}$. Now extend the automorphism g' of $\mathbb{L}$ to $\widehat{g} \in \mathrm{Aut}(\mathbb{C})$.

Combining these facts we see that $X_{\mathbb{C}}$ and $X'_{\mathbb{C}}$ are conjugate via $\widehat{g}$ and thus are the desired non-isomorphic conjugate derived equivalent K3 surfaces.

5. Abelian surfaces

We will now show that a similar result holds for abelian surfaces.
First, recall that for any algebraically closed field $\mathbb{K}$ of characteristic 0 there exists a coarse moduli space $\mathcal{A}_{(1,t)}$ for polarized abelian surfaces of type $(1,t)$. This moduli space is a quasi-projective normal threefold.
Using [3] we know that there exists an involution f on $\mathcal{A}_{(1,t)}$ sending a polarized abelian surface (A,L) to the dual polarized abelian surface $(\widehat{A},\widehat{L})$ (thus Prop. 3.1 is known for abelian surfaces). Here, the dual polarization is e.g. defined by demanding that its pullback under the isogeny induced by L is tL. For other descriptions of $\widehat{L}$ (in the case where the ground field is $\mathbb{C}$) see [2].
In the case $\mathbb{K}=\mathbb{C}$ we use [6, Theorem 3.4] to deduce that the fixed point locus of $f_{\mathbb{C}}$ contains at least one divisor (the points of this divisor correspond to so called Humbert surfaces). Same arguments as in the K3 case show that $\mathrm{Fix}(f)$ contains a divisor as well. We can then proceed as in section 4 to produce a certain curve in $\mathcal{A}_{(1,t)}$. To conclude we just note that by the classical results of Mukai, see [13], A and $\widehat{A}$ are always derived equivalent.

6. Appendix

In this section we will give a different proof of Proposition 3.1. The strategy is as follows: First prove that any Mukai involution is an analytic morphism of the moduli space over $\mathbb{C}$, then show that it is in fact algebraic and using this conclude that a Mukai involution over $\overline{\mathbb{Q}}$ is a morphism as well.

Proposition 6.1. *Any Mukai involution is an automorphism of $\mathcal{M}_{2d}^{\mathbb{C}}$ considered as a complex-analytic space.*

Proof. Let $g_{\mathbb{C}}$ be an arbitrary Mukai involution sending X to $M(r,l,s) =: M(v)$. We know from [14] that there exists a Hodge isometry $\alpha : \Lambda \simeq v^{\perp}/\mathbb{Z}v$ (where $v^{\perp}$ is the orthogonal complement of v in the Mukai lattice) and α clearly induces an isomorphism $\mathbb{P}(\Lambda\otimes\mathbb{C}) \simeq \mathbb{P}((v^{\perp}/\mathbb{Z}v)\otimes\mathbb{C})$. This isomorphism restricts to an isomorphism of the period domains Q and Q' on both sides.
Now if X is a K3 surface then we get its associated period point $x_0 \in Q$ by choosing an isometry $\varphi : H^2(X,\mathbb{Z}) \simeq \Lambda$. We get the period point $\widetilde{x_0}$ of $M(v)$ in Q by using the isometry $\psi\circ f^{-1} : H^2(M(v),\mathbb{Z}) \simeq \Lambda$ where $\psi : v^{\perp}/\mathbb{Z}v \simeq H^2(M(v),\mathbb{Z})$ is the isometry from [14]. The map $g_{\mathbb{C}}$ sends x_0 to x_0'. Now $\alpha(x_0)$ is by definition the period point of $M(v)$ considered as a point in Q'. Thus, on the level of period domains the Mukai involution $g_{\mathbb{C}}$ is just the isomorphism f. Factoring out the markings is compatible with this process. Hence the result. $\square$

Proposition 6.2. *A Mukai involution $g_{\mathbb{C}}$ is an algebraic morphism.*

Proof. We want to apply the following theorem of Borel (see [4]):
If Y is a quasi-projective variety and $f : Y \longrightarrow \Omega/\Gamma$ is a holomorphic map to the quotient of a homogeneous space Ω by an arithmetic torsion free group Γ, then f is algebraic.
We wish to apply the theorem to $Y = \mathcal{M}_{2d}^{\mathbb{C}}$, $\Omega = Q_{2d}$, $\Gamma = \Gamma_{2d}$ and $f = g_{\mathbb{C}}$. Since Γ_{2d} is in general not torsion free we cannot apply it directly. However, using results in [18] (compare also [7] and [22]) we can avoid this problem by using level covers and considering the algebraic construction of $\mathcal{M}_{2d}^{\mathbb{C}}$ described previously. Denote the above mentioned open subset of the Hilbert scheme by U.
For $l \geq 3$ consider $\Gamma_{2d}(l)$,the l-th congruence subgroup of Γ_{2d}. This group is torsion free and the projection $Q_{2d}/\Gamma_{2d}(l) \longrightarrow Q_{2d}/\Gamma_{2d}$ is finite. Since the group of automorphisms of a polarized K3 surface that fix the polarization is finite and this group acts faithfully on the cohomology of a K3 surface, we can apply Proposition 2.17 in [18] which gives us a finite Galois covering U' of U such that $U' \longrightarrow U$ has Galois group $\Gamma_{2d}/\Gamma_{2d}(l)$. Thus we get a commutative diagram

$$\begin{array}{ccc} U'/PGL & \longrightarrow & Q_{2d}/\Gamma_{2d}(l) \\ \downarrow & & \downarrow \\ \mathcal{M}_{2d}^{\mathbb{C}} & \xrightarrow{g_{\mathbb{C}}} & Q_{2d}/\Gamma_{2d} \end{array}$$

All the varieties in the diagram are quasi-projective and the vertical arrows are finite and surjective maps given as quotients of the action of a finite group. By Borel's result the upper map is algebraic and thus so is $g_{\mathbb{C}}$. □

To conclude one has to check that g is also a morphism. Since we are working over algebraically closed fields we can switch to the classical language and only consider closed points. So, g is a map of sets and by the above $g_{\mathbb{C}}$ is a morphism, i.e. locally on affine sets given by polynomials which a priori could have non-algebraic coefficients. However, we know that applied to points with $\overline{\mathbb{Q}}$-entries these polynomials give algebraic values.

Lemma 6.3. *Let $X \subset \mathbb{A}^n_{\overline{\mathbb{Q}}}$ be an affine variety with coordinate ring $K[X]$ and let $p \in K[X_{\mathbb{C}}]$ be a function having algebraic values on X. Then $p \in K[X]$.*

Proof. Set

$$G := \left\{ \sigma \in \mathrm{Aut}(\mathbb{C}) \ : \ \sigma_{|\overline{\mathbb{Q}}} = \mathrm{id} \right\},$$

then $\mathbb{C}^G := \{c \in \mathbb{C} \ : \ \sigma(c) = c \ \forall\, \sigma \in G\} = \overline{\mathbb{Q}}$. It follows that $\mathbb{C}[x_1, \ldots, x_n]^G = \overline{\mathbb{Q}}[x_1, \ldots, x_n]$ and $K[X_{\mathbb{C}}]^G = K[X]$. Now consider $p - gp$ for a polynomial as above and a $g \in G$. By assumption we have that $p - gp$ is zero on $X_{\mathbb{C}}(\overline{\mathbb{Q}})$ and therefore also on $X_{\mathbb{C}}(\mathbb{C})$ since $X_{\mathbb{C}}(\overline{\mathbb{Q}}) \subset X_{\mathbb{C}}(\mathbb{C})$ is dense. Therefore $p - gp = 0 \in K[X_{\mathbb{C}}]$ and thus $p \in K[X]$. □

Thus we have proved the

Proposition 6.4. *The map g is a morphism.* □

References

[1] *Géométrie des surfaces K3: modules et périodes*, Séminaire Palaiseau, ed A. Beauville, J.-P. Bourguignon, M. Demazure, Astérisque 126 (1985)

[2] C. Birkenhake, H. Lange *The dual polarization of an abelian variety*, Arch. Math. 73, 380–389 (1999)

[3] C. Birkenhake, H. Lange *An isomorphism between moduli spaces of abelian varieties*, Math. Nachr. 253, 3–7 (2003)

[4] A. Borel *Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem*, J. Diff. Geom. 6 (1972), 543–560

[5] S. Bosch, W. Lütkebohmert, M. Raynaud *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer 1990

[6] V. Gritsenko, K. Hulek *Minimal Siegel modular threefolds*, Math. Proc. Cambridge Philos. Soc., 123, 461–485 (1998)

[7] V. Gritsenko, K. Hulek, G.K. Sankaran *Moduli spaces of irreducible symplectic manifolds*, MPI Preprint 2008–20

[8] A. Grothendieck *Revêtements etales et groupe fondamental*, Springer-Verlag Berlin 1971 Séminaire de Géométrie Algébrique du Bois Marie 1960-1961 (SGA 1), Dirigé par Alexandre Grothendieck. Augmenté de deux exposés de M. Raynaud, Lecture Notes in Math., Vol. 224

[9] R. Hartshorne *Algebraic Geometry*, Springer, 1977

[10] D. Huybrechts *Generalized Calabi-Yau structures, K3 surfaces, and B-fields*, Int. J. Math. 16 (2005), 13–36

[11] D. Huybrechts, M. Lehn *The geometry of moduli spaces of sheaves*, Aspects Math. 31, Vieweg, Braunschweig, Germany, 1997

[12] D. Morrison *On K3 surfaces with large Picard number*, Invent. Math. 75 (1984), 105–121

[13] S. Mukai *Duality between $D(X)$ and $D(\widehat{X})$ and its application to Picard sheaves*, Nagoya Math. J. 81 (1981), 153–175

[14] S. Mukai *On the moduli space of bundles on K3 surfaces, I*, In: Vector bundles on algebraic varieties, Bombay, (1984), 341–413

[15] D. Mumford *Abelian varieties*, Oxford University Press (1970)

[16] D. Mumford, J. Fogarty *Geometric Invariant Theory*, Erg. Math. 34, 2nd ed., Springer, 1982

[17] D. Orlov *Equivalences of derived categories and K3 surfaces*, J. Math. Sci. 84 (1997), 1361–1381

[18] H. Popp *On moduli of algebraic varieties I*, Inventiones math. 22 (1973), 1–40

[19] B. Saint-Donat *Projective models of K-3 surfaces*, Amer. Jour. of Math. 96 (1974), no.4, 602–639

[20] T. Shioda, H. Inose *On singular K3 surfaces*, Complex analysis and algebraic geomerty: papers dedicated to K. Kodaira. Iwanami Shoten and Cambridge University Peress 1977, 119–136

[21] P. Stellari *A finite group acting on the moduli space of K3 surfaces*, Trans. Amer. Math. Soc. 360 (2008), 6631–6642

[22] B. Szendrői *Some finiteness results for Calabi-Yau threefolds*, J. London Math. Soc. 60 (1999), 689–699

[23] M. Tousi, S. Yassemi *Tensor product of some special rings*, J. Algebra 268 (2003), 672–676

[24] E. Viehweg *Quasi-projective moduli for polarized mainifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 30, Springer, 1995

[25] P.B. Yale *Automorphisms of the complex numbers*, Mathematics Magazine Vol. 39, No. 3. (May, 1966), pp. 135–141

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